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Anomalous diffusion without scale invariance

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Abstract
Asymptotic behaviour of a new class of anomalous diffusion equations for subdiffusive transport defined in terms of generalized distributed fractional-order time derivatives is considered. The effect of slowly varying factors on the scaling function of asymptotic solutions is demonstrated. The origin of slowly varying scaling factors in the CTRW models is discussed.

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Notation

$\mathrm{D}f(x) := \mathrm{d}f(x)/\mathrm{d}x$;
$\mathcal{F}(f) := \hat{f}$; $\quad \hat{f}(k) := \int \mathrm{e}^{\mathrm{i}k\cdot x} f(x)\,\mathrm{d}x$;
$\hat{F}(p,k) := \int_0^\infty \mathrm{e}^{-pt}\left[\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}k\cdot x} F(t,x)\,\mathrm{d}x\right]\mathrm{d}p$;
$\mathcal{L}(f) := \tilde{f}$; $\quad \tilde{f}(p) := \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t$;
$f \sim_y g$ if $g(x) > 0$ and $\lim_{x\to y} f(x)/g(x) = 1$.

1. Introduction

Anomalous diffusion is commonly modelled by scale-invariant equations. This restriction can be traced back to a simplified view of asymptotics of waiting times in the continuous time random walk (CTRW) models which assumes an exact power law behaviour (and stable Lévy probability densities) while ignoring unbounded slowly varying factors such as logarithms. However recent work has provided examples of logarithmic growth of mean square displacement [1] (ultra-slow kinetics). In the framework of CTRW, a logarithmic decay of cumulative waiting time distribution is considered in [2]. In some rigorous stochastic models of CTRW (e.g., see [3]), slowly varying factors in the scaling functions disappear in the process of passing from an integer-valued counting process to a continuous subordinator (the inverse time process). The resulting limitation of scaling functions is an artefact of the construction of the compound stochastic process. A more complicated stochastic model of CTRW for ultra-slow kinetics with a slowly varying scaling factor has been constructed

[4]. More general distributed-order diffusion equations appear in the context of asymptotics of viscoelastic wave propagation [5]. Distributed-order derivatives provide a mathematical model of diffusion equations with scaling functions which are not power functions.

Distributed-order fractional derivatives constitute a new class of pseudo-differential operators leading to more complex scaling laws. Distributed-order derivatives have found applications in modelling material response in dielectrics, viscoelasticity and dynamics [6–10]. Distributed-order derivatives have also been applied in anomalous diffusion [7], in particular for modelling ultra-slow kinetics [11, 12]. Diffusion equations with distributed-order derivatives with respect to space and time were proposed in [13]. Existence and uniqueness for initial-value problems and multi-point boundary-value problems for distributed-order fractional derivatives were made in [14] and a numerical method for solving such equations was proposed in [15].

In this work, the concept of a distributed-order derivative has been generalized by using more general measures on the interval of fractional orders. The new concept includes fractional multi-derivative equations, ultra-slow kinetics as well as equations with symbols which are regularly varying functions of the variable p corresponding to the time derivative at $p \to 0$. Since the concept of a distributed-order fractional derivative has been reserved for a more restricted class of operators we shall use the term 'generalized distributed-order derivative'. In particular, generalized distributed-order derivatives include multi-term fractional differential operators, studied in [16] as well as distributed-order derivatives.

The asymptotic expressions for the solutions of the signalling problem and the initial-value problem involve the solutions of simpler problems, however, with modified scaling factors. The long-range asymptotics of the solutions of the signalling problem and the large-time asymptotics of the initial-value problem are affected only by the lower end of the spectrum of fractional orders. The scaling factors are regularly varying functions with non-trivial slowly varying factors.

We show that the concept of generalized distributed-order derivatives leads a new class of anomalous diffusion equations with scaling of the type $t/W(x)$, $W(x) \sim_\infty x^{2/\alpha} L(x)$ for large propagation distance x and $x/w(t)$, $w(t) = t^{\alpha/2} l(t)$ for large time t, where $L(x), l(t)$ are slowly varying functions at infinity.

A further generalization of the pseudo-differential operators replacing the time derivative would lead to asymptotic solutions involving a general regularly varying scaling function $W(x) = x^\alpha L(x)$, where L is a slowly varying function: $L(\lambda x)/L(x) \to 1$ for $x \to \infty$ and $\lambda > 0$. The pseudo-differential operators appearing in the corresponding diffusion equations involve regularly varying symbols. CTRW provide a model of this kind of diffusion.

In section 2, the generalized distributed-order diffusion equations are defined. In sections 3 and 4, the solutions and the connections to convolution semigroups are considered. Long-range asymptotics of the signalling problem and large-time asymptotics of the initial-value problems are studied in sections 5–7.

Some technicalities needed for the proofs are explained in the appendices.

2. A diffusion equation with a generalized scaling property

We shall consider the initial-value problem

$$\int_{[\alpha,1]} \mathrm{D}^\beta u(t,x)\,\mathrm{d}h(\beta) = \nabla^2 u(t,x) + F(x), \qquad x \in \mathbb{R}^d, \quad t \geqslant 0 \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d \tag{2}$$

as well as the signalling problem

$$\int_{[\alpha,1]} \mathrm{D}^{\beta} u(t,x)\,\mathrm{d}h(\beta) = \nabla^2 u(t,x) + F(x), \qquad x \geqslant 0, \quad t \geqslant 0 \tag{3}$$

$$u(t,0) = 1, \qquad t \geqslant 0, \tag{4}$$

$0 < \alpha < 1$, where D^{β} denotes the Caputo fractional derivative [17]

$$\mathrm{D}^{\beta} f(t) := \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \mathrm{D} f(s)\,\mathrm{d}s, \qquad 0 < \beta < 1, \tag{5}$$

and h is a right-continuous non-decreasing function on the interval $[\alpha, 1]$ satisfying $h(\beta) = 0$ for $\beta < \alpha$. The distributed-order derivative as defined in [18] corresponds to $\mathrm{d}h(\beta) = f(\beta)\,\mathrm{d}\beta$, while multi-term fractional differential operators correspond to piecewise constant functions h.

Equation (1) is equivalent to the equation

$$\int_0^t \left[\int_{[\alpha,1]} \frac{s^{-\beta}}{\Gamma(1-\beta)}\,\mathrm{d}h(\beta) \right] \mathrm{D}u(t-s,x)\,\mathrm{d}s = \nabla^2 u(t,x) + F(x). \tag{6}$$

We shall consider mild solutions of equations (1) and (2). Mild solutions of (1) and (2) are defined as the solutions of the Laplace-transformed equation

$$g(p)[\tilde{u}(p,x) - u(0+,x)/p] = \nabla^2 \tilde{u}(p,x) + \frac{F(x)}{p}, \tag{7}$$

where

$$\tilde{u}(p,x) := \int_0^\infty \mathrm{e}^{-pt} u(t,x)\,\mathrm{d}x \tag{8}$$

and

$$g(p) := \int_{[\alpha,1]} p^{\beta}\,\mathrm{d}h(\beta). \tag{9}$$

Since h has at most a countable number of jump discontinuity points a_n on $[\alpha, 1]$,

$$g(p) = p^{\alpha} \left[\sum_{n=1}^{\infty} p^{a_n} \Delta_n + \sum_{n=0}^{\infty} p^{a_n} \int_{[a_n, a_{n+1}]} p^{\gamma}\,\mathrm{d}h_n(\gamma) \right], \tag{10}$$

where $a_0 = \alpha$, $\Delta_n = \lim_{\varepsilon\to 0}[h(a_n+\varepsilon) - h(a_n-\varepsilon)]$ and the functions h_n are continuous non-decreasing and satisfy the normalization condition $h_n(a_n) = 0$. If $\mathrm{d}h_n(\beta) = \psi_n(\beta)\,\mathrm{d}\beta$, $n = 0, 1, \ldots$, then the operator $\int_{[\alpha,1]} \mathrm{D}^{\beta}\,\mathrm{d}h(\beta)$ is a sum of products of fractional derivative operators and distributed-order fractional derivative operators.

3. Solutions of the initial-value problem

The one-dimensional solution of equations (1) and (2) with $u_0 = U_0\delta(x)$, $F(x) = F_0\delta(x)$ is given by the equation

$$u(t,x) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}} \frac{\mathrm{d}p}{p}\,\mathrm{e}^{pt} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2 + g(p)} [U_0 g(p) + F_0]\,\mathrm{d}k, \tag{11}$$

where the Bromwich contour $\mathcal{B}$ runs upwards along the imaginary axis in the complex p-plane with a detour to the right of $p = 0$.

Note that $\operatorname{Re} g(\mathrm{i}\omega) = \int_{[\alpha,1]} |\omega|^{\beta} \cos(\pi\beta/2)\,\mathrm{d}h(\beta) > 0$ for $\omega \in \mathbb{R}$. Choose the square root in such a way that $\operatorname{Re} g(p)^{1/2} > 0$. Closing the contour of the inner integral by a half-circle in the upper half of the complex k-plane if $x > 0$ and in the lower half k-plane if $x < 0$, the residue calculus yields

$$u(t,x) = U^{(1)}(t,|x|) \tag{12}$$

$$U^{(1)}(t,r) := \frac{1}{4\pi \mathrm{i}} \int_{\mathcal{B}} \frac{1}{g(p)^{1/2}}\, \mathrm{e}^{pt - g(p)^{1/2} r} [U_0 g(p) + F_0] \frac{\mathrm{d}p}{p}. \tag{13}$$

The three-dimensional Green's functions can be obtained from the one-dimensional Green's functions (12) by a simple differentiation [19]:

$$u^{(3)}(t,x) = U^{(3)}(t,|x|), \qquad x \in \mathbb{R}^3 \tag{14}$$

$$U^{(3)}(t,r) := -\frac{1}{2\pi r}\frac{\partial}{\partial r} U^{(1)}(t,r). \tag{15}$$

Equation (14) is a special case of a more general relation

$$G^{(d+2)}(t,r) = -\frac{1}{2\pi r}\frac{\partial}{\partial r} G^{(d)}(t,r), \tag{16}$$

which holds for even and odd dimensions [20].

4. Distributed derivatives, convolution semigroups and infinitely divisible probability densities

Definition 4.1. *A non-negative function $f \in \mathcal{C}^\infty(\overline{\mathbb{R}_+}; \mathbb{R})$ is a Bernstein function if its derivatives satisfy the equations $(-1)^n \mathrm{D}^n f(x) \leqslant 0$ for $n \in \mathbb{Z}_+$.*

Since

$$g(p) = \int_{[\alpha,1[} p^\beta \,\mathrm{d}h(\beta) + bp \tag{17}$$

and $b = h(1) - h(1-) \geqslant 0$, g is a Bernstein function.

Assume that g is a Bernstein function satisfying $g(0) = 0$. This condition is satisfied if either $\alpha > 0$ or $h(0) = h(0-)$. By a theorem on the integral representation of Bernstein functions ([21], theorem 2.9.8)

$$g(p) = ap + \int_{\{0,\infty\{} [1 - \mathrm{e}^{-\lambda p}]\, m(\mathrm{d}\lambda),$$

where m is a positive Radon measure satisfying the condition

$$\int_{]0,\infty\{} \frac{\lambda}{1+\lambda} m(\mathrm{d}\lambda) < \infty. \tag{18}$$

Furthermore $\mathrm{e}^{-g(p)}$ is the Laplace transform of an infinitely divisible probability density and m is its Lévy measure. It follows that the Laplace-domain multiplication $\tilde{f}(p) \to \mathrm{e}^{-sg(p)} \tilde{f}(p)$ defines a strongly continuous semigroup $\{T_s\}_{s\geqslant 0}$ preserving positivity and normalization of probability densities f [21].

We shall now determine the measure m and the constant a.

Since $[1 - \mathrm{e}^{-x}]/x \leqslant \mathrm{e}^{-\vartheta x} \leqslant 1$ $(0 \leqslant \vartheta \leqslant 1)$, the difference quotient

$$[\mathrm{e}^{-\lambda p} - \mathrm{e}^{-\lambda(p+\Delta)}]/\Delta$$

is majorized by the function $\lambda\,\mathrm{e}^{-\lambda p}$ integrable with respect to $m(\mathrm{d}\lambda)$ over $]0,\infty[$. Hence

$$g'(p) = a + \int_{]0,\infty[} \lambda\,\mathrm{e}^{-\lambda p} m(\mathrm{d}\lambda).$$

Equation (17) implies $\lim_{p\to\infty} g'(p) = b$. On the other hand, equation (18) and the obvious inequality $\mathrm{e}^x/(1+x) \geqslant 1$ imply that

$$\int_{]0,\infty[} \lambda\,\mathrm{e}^{-\lambda} m(\mathrm{d}\lambda) < \infty.$$

Hence, by the Lebesgue-dominated convergence theorem, $a = \lim_{p\to\infty} g'(p) = b$.

The Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda p}\lambda m(\mathrm{d}\lambda) = \int_{[\alpha,1[} \beta p^{\beta-1}\,\mathrm{d}h(\beta)$$

can be inverted with the ansatz $\lambda m(\mathrm{d}\lambda) = \phi(\lambda)\,\mathrm{d}\lambda$. Applying the Fubini theorem and a complex contour deformation:

$$\begin{aligned}\phi(\lambda) &= \frac{1}{2\pi\,\mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \left[\int_{[\alpha,1[} \beta p^{\beta-1}\,\mathrm{d}h(\beta)\right] \\ &= \int_{[\alpha,1[} \frac{\beta\sin(\pi(1-\beta))}{\pi} \left[\int_0^\infty \mathrm{e}^{-\lambda r} r^{\beta-1}\,\mathrm{d}r\right] \mathrm{d}h(\beta) \\ &= \int_{[\alpha,1[} \frac{\lambda^{-\beta}}{\Gamma(1-\beta)} \frac{\beta}{\Gamma(\beta)}\,\mathrm{d}h(\beta), \qquad \lambda \geqslant 0\end{aligned}$$

5. Regularly varying analytic functions

Asymptotic analysis of the solution of equation (1) requires some background in regular variation theory.

Definition 5.1 [24]. *An analytic function f defined and non-vanishing in a sector S_γ: $|\arg p| < \gamma$, $|p| > R$, of the complex p-plane is said to be*

(i) slowly varying at ∞ if

$$\lim_{\substack{p\to\infty \\ p\in S_\gamma}} f(\lambda p)/f(p) = 1$$

uniformly with respect to $|\arg p| \leqslant \rho$ for every $\rho < \gamma$ for all $\lambda > 0$;

(ii) regularly varying at ∞ if the limit of $f(\lambda p)/f(p)$ as $p\to\infty$ in S_γ is finite for all $\lambda > 0$.

Theorem 5.1 [24]. *If f is regularly varying at ∞ then there is a function $l(p)$ slowly varying at ∞ and a real number α such that $f(p) = p^\alpha l(p)$.*

The real number α is called the index of the function f at ∞. The class of all the analytic functions in S_γ regularly varying with index α is denoted by $\mathrm{RV}^\alpha_{\infty,\gamma}$. $\mathrm{SV}_{\infty,\gamma} := \mathrm{RV}^0_{\infty,\gamma}$ is the class of non-vanishing analytic functions in S_γ slowly varying at infinity. The class $\mathrm{RV}^\alpha_{0,\gamma}$ consists of all the analytic functions $z \to f(1/z)$ such that $f \in \mathrm{RV}^{-\alpha}_{\infty,\gamma}$ and $\mathrm{SV}_{0,\gamma} := \mathrm{RV}^0_{0,\gamma}$.

Definition 5.2. *An analytic function $f(p)$ defined and not vanishing anywhere in S_R: $|\arg p| < \gamma$, $|p| < R$, $R, \gamma > 0$, is said to be*

(i) slowly varying at 0 if $z \to f(1/z)$ is slowly varying at ∞;

(ii) regularly varying at 0 with index α if $z \to f(1/z)$ is regularly varying at ∞ with index $-\alpha$.

The notion of a regularly varying real function defined on the real line is discussed in more detail in [25, 26]. In the case of real functions we shall ignore the parameter γ.

Let $H(\xi) := h(\xi + \alpha)$.

Proposition 5.2. *If $H \in \mathrm{RV}^{\rho}_{0,\gamma}$, $\rho > 0$, and, more specifically,*

$$H(\xi) = \xi^{\rho} L(\xi)$$

with L slowly varying at 0, then g is analytic in a sector S with $\gamma > \pi/2$. Furthermore,

$$g(p) = p^{\alpha} |\ln p|^{-\rho} L(1/|\ln p|).$$

Proof. By the Karamata theorem on the Laplace transforms of regularly varying functions [26]

$$g(p) = p^{\alpha} \int_{[0,1-\alpha]} \mathrm{e}^{\xi \ln p} \, \mathrm{d}h(\xi) = p^{\alpha} |\ln p|^{-\rho} L(1/|\ln p|).$$

□

Corollary 5.3. *Under the hypotheses of proposition 5.2 $g(p) = p^{\alpha} l(p)$, where l is slowly varying at 0.*

6. Long-range asymptotics of the solutions of the signalling problem for the generalized wave-diffusion equation

In order to get a better insight into the solution we now consider the asymptotic behaviour of the solution of equations (1) and (2) with $F(x) \equiv 0$, $u_0 = U_0 \delta$ for $|x| \to \infty$.

The solution of the signalling problem (3), (4) is given by the equation

$$u(t, x) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}} \mathrm{e}^{pt - xg(p)^{1/2}} \frac{\mathrm{d}p}{p}, \qquad x \geqslant 0. \tag{19}$$

Substitute $p = z/w$ into equation (12). Consider the term in the exponent $Q := g(p)^{1/2}|x| = |x| z^{\alpha/2} l(z/w)^{1/2}/w^{\alpha/2}$ for large $w \in \mathbb{R}_+$. Since l is slowly varying at 0, we shall eliminate it so that Q is replaced by $Q_1 = |x| z^{\alpha/2}/q(w)$, where

$$q(w) := w^{\alpha/2}/l(1/w)^{1/2}. \tag{20}$$

We shall then solve the equation $q(w) = r$ for large w, to obtain a function $W(r)$ such that $q(W(r)) \sim_{\infty} r$. Using the scaling function W we can eliminate $q(w)$ by substituting $w = W(|x|)$. A transformation of the integration variable will yield a function of $t/W(|x|)$. We shall show that the $\mathcal{L}^2$ error introduced by the two consecutive simplifications of equation (19) tends to 0 as $|x| \to 0$. The scaling function W defined by the equation

$$W(r) := \inf\{w \in [0, \infty[\mid q(w) > r\} \tag{21}$$

is non-decreasing. Since $q \in \mathrm{RV}^{\alpha}_{\infty,\gamma}$, by theorem 1.5.12 in [26] $W \in \mathrm{RV}^{1/\alpha}_{\infty}$ and $q(W(r))/r \to 1$ as $r \to \infty$. Substituting $g(p) = p^{\alpha} l(p)$, $l \in \mathrm{SV}_0$, $0 < \alpha \leqslant 1$, $p = z/w$, $l(z/w) \sim l(1/w)$ and defining W by equation (21) we get the asymptotic behaviour as described in the following theorem:

Theorem 6.1. *If $g(p) = p^{\alpha} l(p)$, $l \in \mathrm{SV}_0$, $0 < \alpha \leqslant 1$, $l(p) > 0$ for $p > 0$, then*

$$\|u(\cdot, x) - U(\cdot, x)\|_{\mathcal{L}^2(\mathbb{R}_+;\mathbb{R})} \to 0 \qquad \text{for} \quad x \to \infty, \tag{22}$$

where

$$U(t,x) = P_{\alpha/2}(t/W(x)), \tag{23}$$

$W(x) = x^{2/\alpha}L_1(x)$, $L_1 \in \mathrm{SV}_\infty$ *and* P_γ *is the one-sided* γ*-stable Lévy probability cumulative distribution function,* $0 < \gamma < 1$.

The asymptotic behaviour of the function L_1 can sometimes be expressed in terms of l. If l is a product of powers of logarithm and repeated logarithms then propositions 1.5.15 and 2.3.5 of [26], reported in appendix C, can be applied. Let $l_1(w^\alpha) := 1/l(1/w)$. Clearly $l_1 \in \mathrm{SV}_\infty$. Theorem C.2 shows that $W(x) \sim_\infty x^{2/\alpha} l_1^\sharp(x^2)^{1/\alpha}$, where $l_1^\sharp$ denotes the de Bruijn conjugate of l_1 (definition C.1). By theorem C.3, $l_1^\sharp \sim_\infty 1/l_1$, hence ultimately

$$W(x) \sim_\infty x^{2/\alpha}/l_1(x^2)^{1/\alpha} = x^{2/\alpha} l(1/x^{2/\alpha})^{1/\alpha}. \tag{24}$$

Equation (24) holds in particular for $g(p)$ given by equation (10) with a regularly varying h.

7. Large-time asymptotics of the solutions of the initial-value problem

Long-time asymptotic solutions of the Cauchy problem will be derived from theorem 4.2 in [27]. Let $b(p) := 1/g(p)$, $p = r\,\mathrm{e}^{\mathrm{i}\phi}$, $0 \leqslant \phi \leqslant \pi$. Since

$$\mathrm{Im}\frac{g(p)}{p} = \int_{[\alpha,1]} r^{\beta-1}\sin((\beta-1)\phi)\,\mathrm{d}\beta \leqslant 0$$

and $g(p)/p$ tends to 0 as $p \to \infty$ along the real axis, the function $g(p)/p$ is the Laplace transform of a completely monotone function [28]. By theorem D.3, the function $b(p)/p$ is the Laplace transform of a Bernstein function [21] and $b(p)$ is the Laplace transform of a completely monotone locally integrable function $a(t)$. Let $A(t) := \int_0^t a(s)\,\mathrm{d}s$. If $g(p) = p^\alpha l(p)$, $l \in \mathrm{SV}_0$, $l(p) > 0$ for $p > 0$ and $0 < \alpha < 1$, then $b(p) = p^{-\alpha}/l(p)$. Therefore, by the Karamata Tauberian theorem (theorem 1.7.1 in [26]) $A(t) \sim_\infty t^\alpha/[l(1/t)\Gamma(1+\alpha)]$. But the function A is non-negative and monotone, hence $a(t) \sim_\infty t^{\alpha-1}/[l(1/t)\Gamma(\alpha)]$.

Since a is locally integrable and completely monotone, it is of positive type in the sense of [28] (appendix D) and therefore, by theorem D.2,

$$\lim_{\varepsilon\to 0+} \mathrm{Re}\,\tilde{A}(\mathrm{i}\omega+\epsilon) \geqslant 0 \qquad \text{for all} \quad \omega \in \mathbb{R}$$

Furthermore, $A \in \mathrm{RV}_\infty^{\alpha-1}$ with $0 < \alpha < 1$ by the monotone density theorem (theorem B.1). The following theorem can be found in [27] (theorem 4.2),

Theorem 7.1. *If (1)* $\liminf_{\epsilon\to 0}[a + \mathrm{Re}\,\tilde{f}(\mathrm{i}\omega+\epsilon)] \geqslant 0$ *for* $\omega \in \mathbb{R}$*; (2) the function* $F(t) = a + \int_0^t f(s)\,\mathrm{d}s$ *is positive for* $t > T$*, for some* $T > 0$*, and* $F \in \mathrm{RV}_\infty^\beta$ *with* $-1 < \beta \leqslant 1$*, (3)* u *is the solution of the equation*

$$u_t = a_0\nabla^2 u + f * \nabla^2 u$$

for $x \in \mathbb{R}^d$*, then the scaled solution* $w(T)^d u(Tt, w(T)x)$ *with* $w(T) = [tF(t)\Gamma(\beta+1)]^{1/2}$ *tends to* $g(t,x)\int u(0,x)\,\mathrm{d}x$*, where* $g(t,x) = \mathcal{F}^{-1}(E_{\beta+1}(-|k|^2 t^{\beta-1})$*, in every Sobolev space* $H^{-s}(\mathbb{R}^d)$ *such that* $s > d/2$.

We recall that the Sobolev space H^{-s} is the space of tempered distributions g whose Fourier transform has a finite norm [29]

$$\|g\|_{H^{-s}} := \left[\int_{\mathbb{R}^d}(1+k^2)^{-s}|\hat{g}(k)|^2\,\mathrm{d}k\right]^{1/2}.$$

By theorem 7.1, the scaled solution

$$w(T)^d u(Tt, w(T)x)$$

of equations (1) and (2), with the scaling function

$$w(t) = t^{\alpha/2}/l(1/t)^{1/2}, \tag{25}$$

tends in H^{-s}, $s > d/2$, to

$$V^{(d)}(t,x) := U_0 \mathcal{F}^{-1}(E_\alpha(-|k|^2 t^\alpha) \tag{26}$$

as $T \to \infty$, where E_α denotes the Mittag–Leffler function [30] and $U_0 = \int u_0(x)\,\mathrm{d}x$. Since $f(k) := E_\alpha(-|k|^2 t^\alpha) \sim |k|^{-2} t^{-\alpha}/\Gamma(1-\alpha)$ for $|k| \to \infty$ and $E_\alpha(0) = 1$, the inverse Fourier transform of f belongs to $H^{-s}(\mathbb{R}^d; \mathbb{R})$ for $s > d/2 - 2$.

In the one-dimensional case

$$V^{(1)}(t,x) = \frac{U_0}{2} t^{-\alpha/2} M_{\alpha/2}(|x|/t^{\alpha/2}), \tag{27}$$

where the function $M_\gamma, 0 < \gamma < 1$, is a special case of the Wright function W: $M_\gamma(z) = W_{-\gamma,1-\gamma}(-z)$ [19, 31]. Equation (27) can be obtained from equation (26) by transforming the inverse Fourier transform to a Laplace transform by means of the identity (e.g., [17], equation (1.95))

$$E_{2\gamma}(z) = \tfrac{1}{2}[E_\gamma(z^{1/2}) + E_\gamma(-z^{1/2})], \qquad -\pi < \arg z < \pi,$$

with $z^{1/2} = \mathrm{i}k$, and then taking into account the Laplace transform pair $W_{-\mu,\nu}(-t) \div E_{\mu,\mu+\nu}(-p)$, valid for $0 < \mu < 1$ [32, 33].

The function M_β decays very rapidly [31]

$$M_\beta(y) \sim \frac{1}{\sqrt{2\pi}(1-\beta)^\beta \beta^{2\beta-1}} Y^{\beta-1/2}\,\mathrm{e}^{-Y}, \qquad y \to \infty \tag{28}$$

$$Y := (1-\beta)[\beta^\beta \quad y]^{1/(1-\beta)} \tag{29}$$

while it is finite at 0

$$M_\beta(0) = 1/\Gamma(1-\beta),$$

and the probability density $u(t,\cdot)$ has a finite second-order moment. Since

$$\int w(T)^d u(Tt, w(T)x) x^2\,\mathrm{d}x \to \int U^{(1)}(t,x) x^2\,\mathrm{d}x = ct^\alpha$$

as $T \to \infty$, where c is a positive number,

$$\int u(tT, y) y^2\,\mathrm{d}y \sim c(tT)^\alpha / l(1/T) \qquad T \to \infty.$$

Substituting $t = 1$ and replacing T by t leads to the asymptotic estimate of the mean square displacement

$$\langle r^2 \rangle \sim ct^\alpha / l(1/t), \tag{30}$$

which deviates from the power law.

The three-dimensional asymptotic solution is given by the formula (cf [19])

$$U^{(3)}(t,x) = -\frac{U_0}{2\pi r}\frac{\partial f(t,r)}{\partial r} = -\frac{U_0 t^{-\alpha}}{4\pi r} M'_{\alpha/2}\left(\frac{r}{t^{\alpha/2}}\right). \tag{31}$$

The derivative M'_β decays fast and assumes a finite value $-1/\Gamma(1-2\beta)$ at 0. Consequently, $U^{(d)} \in \mathcal{L}^q(\mathbb{R}^d;\mathbb{R})$ for $d = 1, 3$ and for arbitrary $q > 0$. By theorem 4.2 and proposition 2.1 in [27],

$$\|u(t,\cdot) - v^{(d)}(t,\cdot)\|_{\mathcal{L}^q(\mathbb{R}^d;\mathbb{R})} = \mathrm{o}[t^{-d\alpha(q-1)/2q}], \tag{32}$$

where

$$v^{(1)}(t,x) = \frac{U_0}{2} w(t)^{-1} M_{\alpha/2}(|x|/w(t)) \tag{33}$$

$$v^{(3)}(t,x) = -\frac{U_0}{4\pi} w(t)^{-3} \frac{w(t)}{|x|} M'_{\alpha/2}(|x|/w(t)) \tag{34}$$

uniformly for $t \in [a,b], 0 < a < b < \infty$. The mean square displacement $\langle r^2\rangle \sim w(t)^2$ as $t \to \infty$ in both cases.

The asymptotic solution of the Cauchy problem for arbitrary dimension d is

$$U^{(d)}(t,x) = \frac{U_0\sqrt{\pi}}{(2\pi)^{d/2}|x|^{d/2-1}} \int_0^\infty J_{d/2-1}(\kappa|x|) E_\alpha(-\kappa^2 t^\alpha)\kappa^{d/2}\,\mathrm{d}\kappa. \tag{35}$$

For every dimension d the integral of $U^{(d)}$ is 1.

The Laplace pair

$$E_\gamma(-at^\gamma) \div \frac{p^{\gamma-1}}{a+p^\gamma}$$

implies the asymptotic relation

$$E_\gamma(-at^\gamma) \sim \frac{t^{-\gamma}}{a\Gamma(1-\gamma)}, \qquad t \to \infty.$$

In view of the large-argument asymptotics of the Bessel function the integral is absolutely convergent for $d < 3$. The function $k \to E_\alpha(-|k|^2 t^\alpha)$ is square integrable on $\mathbb{R}^d$ only if $d < 4$. Hence, we do not expect to obtain a square-integrable function $U^{(d)}$ for $d \geqslant 4$.

Since $J_\nu(x) = \mathrm{O}[x^{-1/2}]$ for $x \to \infty$, the integral in equation (35) is absolutely integrable if $d < 4\alpha - 1$.

If $l(p) \equiv 1$ then the asymptotic limit is the exact solution of the time-fractional diffusion equation $\mathrm{D}^\alpha u = \nabla^2 u$ with the initial data $u_0(x) = U_0\delta(x)$.

8. CTRW models of non-scale-invariant anomalous diffusion

The continuous time random walk (CTRW) model assumes an ensemble of particles (walkers) moving in the space $\mathbb{R}^d$ by a sequence of jumps. The probability density $Q(\tau,\xi)$ in spacetime ($\tau \in \mathbb{R}_+, \xi \in \mathbb{R}^d$) of the walker executing a single jump at a distance ξ in a time interval of duration τ. The 'waiting time' τ can be attributed to the particle being trapped at a site before executing an instantaneous jump, to a finite duration of the jump (e.g., a finite velocity) or to other reasons. The choice of the interpretation is irrelevant for the derivation of the master equation. The term 'waiting time' was suggested by the first interpretation.

According to the Montroll–Weiss master equation for the Laplace–Fourier transform

$$\hat{P}(p,k) = \int_0^\infty \mathrm{e}^{-pt}\left[\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}k\cdot x} P(t,x)\,\mathrm{d}x\right]\mathrm{d}t \tag{36}$$

of the particle density $P(t,x)$ is given in terms of the Laplace transform

$$\tilde{q}(p) = \int_0^\infty \mathrm{e}^{-p\tau} q(\tau)\,\mathrm{d}\tau$$

of the waiting time probability density $q(\tau)$ and the Laplace–Fourier transform $\hat{Q}(p,k)$ of the joint probability density $Q(\tau,\xi)$ of a single jump ξ in a time τ,

$$\hat{P}(p,k) = \frac{1-\tilde{q}(p)}{p}\frac{1}{1-\hat{Q}(p,k)}, \tag{37}$$

where $q(\tau)$ is the probability density for the time interval τ between two consecutive jumps, hence it is the marginal probability density

$$q(\tau) = \int Q(\tau,\xi)\,\mathrm{d}\xi. \tag{38}$$

A careful derivation of the master equation can be found in a recent paper ([34], equation (23)).

It is usually assumed [35, 36] that the probability density q is a stable one-sided Lévy probability. This assumption entails that the scaling functions are power functions. There is, however, no compelling reason for this limitation.

Assuming for simplicity that the waiting time and the jump size are two independent random variables and $\hat{Q}(p,k) = \tilde{q}(p)\hat{w}(k)$, where $w(\xi)$ is the probability density of a jump ξ. Assume in addition that $q(\tau) \sim \tau^{-1-\beta} l(1/\tau)/\Gamma(-\beta), 0 < \beta < 1, l \in \mathrm{SV}_0, \tilde{q}(p) \sim 1 - p^\beta l(p)\ (p \to 0), \hat{w}(k) \sim 1 - \sigma^2 k^2\ (k \to 0)$. The Montroll–Weiss master theorem now implies that

$$[p^\beta l(p) + \sigma^2 k^2]\hat{P}(p,k) = p^{\beta-1} l(p).$$

Let L denote the pseudo-differential operator with the symbol $p^\beta l(p)$. The probability distribution $P(t,x)$ defined as the inverse Laplace-inverse Fourier transform of $\hat{P}(p,k)$ satisfies the equation

$$L P(t,x) = \sigma^2 \nabla^2 P(t,x) + s(t), \tag{39}$$

where

$$s(t) \sim_\infty \frac{t_+^{-\beta} l(1/t)}{\Gamma(1-\beta)}.$$

The operator L is a generalization of the distributed-order fractional derivative considered earlier.

The operator with the symbol $g(p)$ has all the relevant properties of p^β in the theory of anomalous diffusion for coupled CTRW developed in [22, 23], indicating further possible generalizations of anomalous diffusion equations. Thus, the probability of the walker reaching $\mathcal{A} \subset \mathbb{R}^d$ within the interval of time $[0,t]$ is $\int_\mathcal{A}\int_0^\infty \mathrm{d}s\, g(\mathrm{D}) T_s f(t,x)\,\mathrm{d}x$ for some original probability spacetime density f (corresponding to the number of steps $s = 0$) and the Fourier–Laplace transform of the spacetime density is $g(p)/[p(g(p)+k^2)]$.

9. Conclusions

Slowly varying factors in the time-derivative operator of the anomalous diffusion equation affect the scaling function in the asymptotic form of the solution. The asymptotic shape of the distribution function retains the pattern of scale-invariant diffusion, but the scaling function deviates from the power law.

Slowly varying factors in the scaling function also originate from the asymptotic tail behaviour of the probability distribution of the CTRW waiting times.

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Appendix A. Proof of theorem 6.1

Changing the integration variable $p = z/w$ in equation (12), where w is a positive number, yields the formula

$$u(t,x) = \frac{1}{4\pi \mathrm{i}} \int_{\mathcal{B}} \exp(zt/w - g(z/w)^{1/2} x) \frac{\mathrm{d}z}{z}$$
$$\equiv \frac{1}{4\pi \mathrm{i}} \int_{\mathcal{B}} \exp(zt/w - z^{1/2} l(z/w)^{1/2} x/w^{1/2}) \frac{\mathrm{d}z}{z}. \quad \text{(A.1)}$$

Let

$$u_1(t,x;w) := \frac{1}{4\pi \mathrm{i}} \int_{\mathcal{B}} \exp(zt/w - z^{1/2} x/q(w)) \frac{\mathrm{d}p}{p}, \quad \text{(A.2)}$$

where the function $q(w)$ is defined by equation (20). By the Parseval theorem

$$\|u(\cdot,x) - u_1(\cdot,x;w)\|^2_{\mathcal{L}^2} \leqslant \frac{1}{4\pi^2} \left| \int_{\mathcal{B}} \mathrm{e}^{zt/w} \left[\mathrm{e}^{-z^{\alpha/2} l(z/w)^{1/2} x/w^{\alpha/2}} - \mathrm{e}^{-z^{\alpha/2} l(1/w)^{1/2} x/w^{\alpha/2}}\right] \frac{\mathrm{d}z}{z} \right|$$
$$\leqslant \mathrm{e}^{2\varepsilon t/w} \frac{1}{2\pi^2} \int_0^\infty \left| \mathrm{e}^{-(\mathrm{i}y+\varepsilon)^{\alpha/2} l(1/w)^{1/2} x/w^{\alpha/2}} - \mathrm{e}^{-(\mathrm{i}y+\varepsilon)^{\alpha/2} l((\mathrm{i}y+\varepsilon)/w)^{1/2} x/w^{\alpha/2}} \right| \frac{\mathrm{d}y}{y^2+\epsilon^2}$$
$$= \mathrm{e}^{2\varepsilon t/w} \frac{1}{2\pi^2} \int_0^\infty \mathrm{e}^{-(\mathrm{i}y+\varepsilon)^{\alpha/2} l(1/w)^{1/2} x/w^{\alpha/2}}$$
$$\times \left[1 - \mathrm{e}^{-(\mathrm{i}y+\varepsilon)^{\alpha/2} l(1/w)^{1/2} [l((\mathrm{i}y+\varepsilon)/w)^{1/2}/l(1/w)^{1/2} - 1] x/w^{\alpha/2}}\right] \frac{\mathrm{d}y}{y^2+\varepsilon^2}$$

for some $\varepsilon > 0$. Note that $\mathrm{Re}(\mathrm{i}y + \epsilon)^{\alpha/2} \sim y^{\alpha/2}\cos(\pi\alpha/2)$ for large y. Let $w \to \infty$. The last expression tends to 0 by the Lebesgue-dominated convergence theorem, hence $\|u(\cdot,x) - u_1(\cdot,x;w)\|^2_{\mathcal{L}^2} \to 0$.

Substituting $w = W(x)$ into equation (A.2) and taking into account that $W(x) \to \infty$ for $x \to \infty$ and

$$\lim_{x\to\infty} x/q(W(x)) = 1,$$

it follows by the Lebesgue-dominated convergence theorem that

$$\|u(\cdot,x) - u_2(\cdot,x)\|^2_{\mathcal{L}^2} \to 0,$$

where $u_2(t,x) := u_1(t,x;W(x)) = P_{\alpha/2}(t/W(x))$.

Appendix B. Monotone density theorem

Theorem B.1 [26, 37]. *If $\int_0^x f(y)\,\mathrm{d}y \sim_\infty cx^a l(x)$, $a \in \mathbb{R}$, $l \in \mathrm{SV}_\infty$, and f is monotone on $[X,\infty[$ for some $X > 0$, then $f(x) \sim_\infty cax^{a-1} l(x)$.*

Appendix C. de Bruijn conjugate functions

Definition C.1. *If $l \in \mathrm{SV}_\infty$ then the de Bruijn conjugate $l^\sharp \in \mathrm{SV}_\infty$ is a solution of the equations*

$$\lim_{x\to\infty} l(x) l^\sharp(x l(x)) = 1, \qquad \lim_{x\to\infty} l^\sharp(x) l(x l^\sharp(x)) = 1.$$

Theorem C.1 [26].

(i) $l^\sharp$ is defined uniquely up to asymptotic equivalence;
(ii) $l^{\sharp\sharp} \sim l$.

Theorem C.2 ([26], proposition 1.5.15). *Let $a, b > 0$, $l \in \mathrm{SV}_\infty$ and $f(x) \sim_\infty x^{ab} l(x^b)^a$. If g is an asymptotic inverse of f then*

$$g(x) \sim_\infty x^{1/(ab)} l^\sharp(x^{1/a})^{1/b}.$$

Theorem C.3 ([26] proposition 2.3.5 and appendix 5.2). *If*

$$l(x) = \prod_{k=1}^{n} (\ln^k(x))^{a_k},$$

where

$$\ln^k := \underbrace{\ln \circ \ln \cdots \circ \ln}_{k \text{ times}}$$

then $l^\sharp \sim_\infty 1/l$.

Appendix D. Functions of positive type

Definition D.1. *A measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ is of positive type if*

$$\int_{-\infty}^{\infty} \int_0^{\infty} f(s)\phi(t-s)\phi(t)\, \mathrm{d}s\, \mathrm{d}t \geqslant 0$$

for every square-integrable function ϕ with compact support on $\mathbb{R}$.

Theorem D.1 ([28], proposition 16.3.1). *If $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ is non-negative, non-increasing and convex, then f is of positive type.*

Theorem D.2. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is of positive type, then*

$$\liminf_{p\to \mathrm{i}\omega} \mathrm{Re}\, \tilde{f}(p) \geqslant 0.$$

The last theorem is a consequence of theorem 16.2.6 in [28]).

Theorem D.3 ([38], theorem 6; [39]). *If the functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ satisfy the Volterra equation $f * g = t\theta(t)$, then the function $f : \mathbb{R}_+ \to \mathbb{R}$ is completely monotone and $\lim_{t\to\infty} f(t) = a$, $0 < a \leqslant \infty$, and locally integrable if and only if g is a Bernstein function with $\lim_{t\to\infty} g(t) = 1/a$.*

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